

# The Coloring Complex and Cyclic Coloring Complex of a Complete $k$ -Uniform Hypergraph

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## Abstract

In this paper, we study the homology of the coloring complex and the cyclic coloring complex of a complete  $k$ -uniform hypergraph. We show that the coloring complex of a complete  $k$ -uniform hypergraph is shellable, and we determine the rank of its unique nontrivial homology group in terms of its chromatic polynomial. We also show that the dimension of the  $(n - k - 1)^{st}$  homology group of the cyclic coloring complex of a complete  $k$ -uniform hypergraph is given by a binomial coefficient. Further, we discuss a complex whose  $r$ -faces consist of all ordered set partitions  $[B_1, \dots, B_{r+2}]$  where none of the  $B_i$  contain a hyperedge of the complete  $k$ -uniform hypergraph  $H$  and where  $1 \in B_1$ . It is shown that the dimensions of the homology groups of this complex are given by binomial coefficients. As a consequence, this result gives the dimensions of the multilinear parts of the cyclic homology groups of  $\mathbb{C}[x_1, \dots, x_n]/\{x_{i_1} \dots x_{i_k} \mid i_1 \dots i_k \text{ is a hyperedge of } H\}$ .

## 1 Introduction

In this paper, we will study the homology of the coloring complex and the cyclic coloring complex of a complete  $k$ -uniform hypergraph. Throughout the paper, let  $G$  be a simple graph on  $n$  vertices.

Consider  $R = A/I$  where  $A = F[x_S \mid S \subseteq [n]]$ ,  $I$  is the ideal generated by  $\{x_U x_T \mid U \not\subseteq T, T \not\subseteq U\}$ , and  $F$  is a field of characteristic zero. The ideal  $K_G$  is defined to be the ideal generated by the monomials  $x_{X_1}^{e_1} x_{X_2}^{e_2} \dots x_{X_l}^{e_l}$ ,  $e_i > 0$  such that for all  $i$ ,  $1 \leq i \leq l+1$ ,  $Y_i = X_i \setminus X_{i-1}$  does not contain an edge of  $G$  ( $X_0 = \emptyset$  and  $X_{l+1} = [n]$ ). Steingrímsson [12] shows that there is a bijection between the monomials of  $K_G$  of degree  $r$  and colorings of  $G$  with  $r+1$  colors. He thus names  $K_G$  the coloring ideal and notes that the quotient  $R/K_G$  is the face ring of a simplicial complex,  $\Lambda(G)$ , or the coloring complex of  $G$ .

Jonsson [8] extended the work of Steingrímsson and proved that for the case where  $G$  has at least one edge,  $\Lambda(G)$  is a constructible complex. He shows that the  $(n-3)^{rd}$  homology group of  $\Lambda(G)$  is the only nonzero homology group and that the dimension of this group is equal to  $\chi_G(-1) - 1$ , where  $\chi_G(\lambda)$  is the chromatic polynomial of  $G$ .

Crown [5] defined and studied the cyclic coloring complex of a graph, denoted  $\Delta(G)$ . In her paper, she determines the dimensions of the homology groups of  $\Delta(G)$ . She shows that for a connected graph,  $G$ , the dimension of the  $(n-3)^{rd}$  homology group of  $\Delta(G)$  is equal to  $n-2$  plus  $\chi'_G(0)$ , and the dimension of the  $r^{th}$  homology group, for  $r < n-3$ , is given by the binomial coefficient  $\binom{n-1}{r+1}$ . If  $G$  has  $k$  connected components, she shows that the dimension of the  $(n-3)^{rd}$  homology group of  $\Delta(G)$  is equal to  $n - (k+1)$  plus  $\frac{1}{k!} \mid \chi_G^k(0) \mid$ , where  $\chi_G^k(\lambda)$  is the  $k^{th}$  derivative of  $\chi_G(\lambda)$ . For  $r < n-3$ , she gives a formula for the dimension of the  $r^{th}$  homology group of  $\Delta(G)$  in terms of a sum of binomial coefficients and the value  $\frac{1}{k!} \mid \chi_G^k(0) \mid$ .

Let  $H$  be a hypergraph on  $n$  vertices. The coloring complex of a hypergraph,  $\Lambda(H)$  was introduced in Long and Rundell [10], as well as in Breuer, Dall, and Kubitzke [3]. In Long and Rundell [10], the authors extend a result of Hanlon [6] in which he shows that there exists a Hodge decomposition of the unique nontrivial homology group of  $\Lambda(G)$  and that the dimension of the  $j^{th}$  Hodge piece of this decomposition equals the absolute value of the coefficient of  $\lambda^j$  in  $\chi_G(\lambda)$ . Long and Rundell [10] extend this result by showing that the Euler Characteristic of the  $j^{th}$  Hodge subcomplex of  $\Lambda(H)$  is related to the coefficient of  $\lambda^j$  in  $\chi_H(\lambda)$ . They also show that for a class of hypergraphs, which they call star hypergraphs, the coloring complex of the hypergraph is Cohen-Macaulay. In the Breuer, Dall, and Kubitzke [3] paper, the authors show that the  $f$ - and  $h$ - vectors of the

coloring complexes of hypergraphs provide tighter bounds on the coefficients of chromatic polynomials of hypergraphs. They also show that the coloring complex of a hypergraph has a wedge decomposition, and they provide a characterization of hypergraphs having a connected coloring complex.

In this paper, we will study the homology of the coloring complex and the cyclic coloring complex of a complete  $k$ -uniform hypergraph, and we list the main results of the paper below. First we will show:

**Theorem 3.1** *Let  $H$  be the complete  $k$ -uniform hypergraph on  $n$  vertices and let  $k > n/2$ . Then  $\Lambda(H)$  is shellable of dimension  $n - k - 1$ .*

From this theorem, we obtain a basis for the cohomology of  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$  (Corollary 3.3), and we also obtain the following result:

**Corollary 3.2** *If  $H$  is the complete,  $k$ -uniform hypergraph on  $n$  vertices, then the homology of  $\Lambda(H)$  is nonzero only in dimension  $n - k - 1$ , and the dimension of  $H_{n-k-1}(\Lambda(H))$  equals the sum of the absolute values of the coefficients of  $\chi_H(\lambda)$  minus one. Moreover, the dimension of the  $j^{\text{th}}$  Hodge piece in the Hodge decomposition of  $\Lambda(H)$  is*

$$\dim(H_{n-k-1}^{(j)}(\Lambda(H))) = (-1)^{n-k}[\lambda^j](\chi_H(-\lambda) - (-\lambda)^n).$$

We also note that:

**Theorem 3.4** *Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices, and let  $v$  be a vertex of  $H$ . Suppose that the edge set of  $H$  consists of all possible hyperedges of size  $k$  containing the vertex  $v$ . Then  $\Lambda(H)$  is shellable.*

**Corollary 3.5** *Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices, and let  $v$  be a vertex of  $H$ . Suppose that  $H$  consists of all possible hyperedges of size  $k$  containing the vertex  $v$ . Then the homology of  $\Lambda(H)$  is nonzero only in dimension  $n - k - 1$ , and the dimension*

of  $H_{n-k-1}(\Lambda(H))$  equals the sum of the absolute values of the coefficients of  $\chi_H(\lambda)$  minus one. Moreover, the dimension of the  $j^{\text{th}}$  Hodge piece in the Hodge decomposition of  $\Lambda(H)$  is

$$\dim(H_{n-k-1}^{(j)}(\Lambda(H))) = (-1)^{n-k}[\lambda^j](\chi_H(-\lambda) - (-\lambda)^n).$$

As a corollary to Theorem 3.4, we are able to obtain a basis for  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$ , and this result is stated in Corollary 3.6.

In Section 4, we begin our study of the cyclic coloring complex of the complete  $k$ -uniform hypergraph with the result:

**Theorem 4.1** *Let  $H$  be the  $k$ -uniform hypergraph on  $n$  vertices with edge set consisting of all possible hyperedges of size  $k$  containing the vertex 1. Then the dimension of  $HC_r(\Delta(H))$  is nonzero for  $n - k - 1 \geq r \geq -1$  and is given by*

$$\dim(HC_r(\Delta(H))) = \binom{n-1}{r+1}.$$

We can define an action of  $S_{r+2}$  on  $\Delta_r$ . Namely, if  $\sigma \in S_{r+2}$ , then  $\sigma \cdot (B_1, \dots, B_{r+2}) = (B_{\sigma^{-1}(1)}, \dots, B_{\sigma^{-1}(r+2)})$ , and this action then makes  $C_r$  into an  $S_{r+2}$ -module, where  $C_r$  is the vector space over a field of characteristic zero with basis  $\Delta_r$ . Let  $\Delta(E_n)$  denote the cyclic coloring complex of the complete graph with looped edges. Using a result from Crown [5], we will obtain the following result:

**Theorem 4.2** *The  $S_n$ -module structure of  $HC_r(\Delta(E_n))$  is  $S^\lambda$  where  $\lambda = (n - r - 1, 1^{r+1})$ . Moreover, this is the  $S_n$ -module structure of the multilinear part of  $HC_r(\mathbb{C}[x_1, \dots, x_n])$ .*

This theorem will give us the following corollary:

**Corollary 4.3** *Let  $H$  be a hypergraph on  $n$  vertices, let  $v$  be a vertex of  $H$ , and let the hyperedges of  $H$  be all possible subsets of  $[n]$  of size  $k$  that contain vertex  $v$ . Then the  $S_n$ -module structure of  $HC_r(\Delta(H))$  is  $S^\lambda$  where  $\lambda = (n - r - 1, 1^{r+1})$ .*

In order to obtain the result of Theorem 4.6, we will define a complex  $\Delta(H)^C$ , whose  $r$ -faces consist of all ordered set partitions  $[B_1, \dots, B_{r+2}]$ , where none of the  $B_i$  contain a hyperedge of  $H$  and where  $1 \in B_1$ . We compute the dimensions of the homology groups of this complex,  $HC_r(\Delta(H)^C)$ , for  $r \geq n - k$ .

**Theorem 4.4** *Let  $H$  be the complete  $k$ -uniform hypergraph on  $n$  vertices. For  $n - 2 \geq r > n - k$ ,*

$$\dim(HC_r(\Delta(H)^C)) = \binom{n-1}{r+1}$$

and

$$\dim(HC_{n-k}(\Delta(H)^C)) = \binom{n-1}{n-k-1} + \binom{n-1}{n-k+1}.$$

This gives, as a result, the dimensions of the multilinear parts of the cyclic homology groups of  $\mathbb{C}[x_1, \dots, x_n]/\{x_{i_1} \dots x_{i_k} \mid i_1 \dots i_k \text{ is a hyperedge of } H\}$ .

**Corollary 4.5** *For the complete  $k$ -uniform hypergraph on  $n$  vertices,  $H$ , the dimension of the multilinear part of the  $r^{\text{th}}$  cyclic homology group of  $\mathbb{C}[x_1, \dots, x_n]/\{x_{i_1} \dots x_{i_k} \mid i_1 \dots i_k \text{ is a hyperedge of } H\}$  is  $\binom{n-1}{r+1}$  for  $n - k \leq r \leq n - 2$  and  $\binom{n-1}{n-k-1} + \binom{n-1}{n-k+1}$  for  $r = n - k$ .*

**Theorem 4.6** *Let  $H$  be a complete  $k$ -uniform hypergraph. Then*

$$\dim(HC_{n-k-1}(\Delta(H))) = \binom{n}{n-k}.$$

Further, for  $k = n - 1$  and  $k = n - 2$ , we determine the dimensions of  $HC_r(\Delta(H))$  for all  $r$ :

**Theorem 4.7** *Let  $H$  be the complete  $(n - 1)$ -uniform hypergraph on  $n$  vertices. Then*

$$\dim(HC_0(\Delta(H))) = \binom{n}{1} = n$$

and

$$\dim(HC_{-1}(\Delta(H))) = \binom{n}{0} = 1.$$

and

**Theorem 4.8** *Let  $H$  be the complete  $(n - 2)$ -uniform hypergraph on  $n$  vertices. Then for  $-1 \leq r \leq 1$ ,*

$$\dim(HC_r(\Delta(H))) = \binom{n}{r+1}.$$

## 2 Preliminaries

**Definition 2.1.** A hypergraph,  $H$ , is an ordered pair,  $(V, E)$ , where  $V$  is a set of vertices and  $E$  is a set of subsets of  $V$ . A hyperedge of  $H$  is an element of  $E$ . A hypergraph is said to be uniform of rank  $k$ , or  $k$ -uniform, if all of its hyperedges have size  $k$ . A  $k$ -uniform hypergraph is complete if every subset of size  $k$  of  $V$  is a hyperedge of  $H$ .

Throughout this paper,  $H$  will denote a hypergraph whose vertex set  $V$  is  $\{1, \dots, n\}$ . We define the coloring complex of a hypergraph,  $H$ , following the presentation in Jons-son [8].

Let  $(B_1, \dots, B_{r+2})$  be an ordered partition of  $\{1, \dots, n\}$  where at least one of the  $B_i$  contains a hyperedge of  $H$ . Further, let  $\Lambda_r(H)$  be the set of ordered partitions of length  $r + 2$  in which at least one part contains a hyperedge of  $H$ , and let  $V_r$  be the vector space over a field of characteristic zero with basis  $\Lambda_r(H)$ .

**Definition 2.2.** The coloring complex of  $H$ , denoted  $\Lambda(H)$ , has as its  $r$ -faces the elements of the set  $\Lambda_r(H)$  and boundary map  $\delta_r : V_r \rightarrow V_{r-1}$  given by

$$\delta_r((B_1, \dots, B_{r+2})) := \sum_{i=1}^{r+1} (-1)^i (B_1, \dots, B_i \cup B_{i+1}, \dots, B_{r+2}).$$

Notice that  $\delta_{r-1} \circ \delta_r = 0$ . Then:

**Definition 2.3.** The  $r^{th}$  homology group of  $\Lambda(H)$  is  $H_r(\Lambda(H)) = \ker(\delta_r) / \text{im}(\delta_{r+1})$ .

It is worth noting that Hultman [7] defined a complex that includes both Steingrímsson's coloring complex and the coloring complex of a hypergraph as a special case.

In Section 3, we will show that for a complete  $k$ -uniform hypergraph  $H$ ,  $\Lambda(H)$  is shellable, and hence, Cohen-Macaulay. We will thus need the following definitions:

**Definition 2.4.** A simplicial complex  $\Delta$  is Cohen-Macaulay over a ring  $R$  if  $\tilde{H}_i(\text{link}_\Delta(\sigma); R) = 0$  for all  $\sigma \in \Delta$  and  $i < \dim(\text{link}_\Delta(\sigma))$ .

**Definition 2.5.** A simplicial complex is pure if all of its maximal faces have the same dimension.

Let  $F$  and  $G$  be sets and suppose  $F \subseteq G$ . The Boolean interval, denoted  $[F, G]$ , is the set  $\{H \mid F \subseteq H \subseteq G\}$ . We let  $\bar{F}$  denote the Boolean interval  $[\emptyset, F]$ .

**Definition 2.6.** A shellable complex,  $\Lambda$ , is a complex whose facets can be ordered  $F_1, \dots, F_m$  so that the subcomplex  $(\bigcup_{i=1}^{l-1} \bar{F}_i) \cap \bar{F}_l$  is pure and has dimension  $(\dim F_l) - 1$  for all  $l = 2, \dots, m$ . If  $\Lambda$  is shellable, the ordering of the facets  $F_1, \dots, F_m$  is called a shelling.

In our proof that  $\Lambda(H)$  is shellable for a complete  $k$ -uniform hypergraph ( $k > n/2$ ), we will use Proposition 2.5 from Björner and Wachs [2] which we include here, along with a necessary definition.

**Definition 2.7.** Let  $F_1, \dots, F_m$  be a shelling of  $\Lambda$ . The restriction of facet  $F_l$ , denoted  $\mathcal{R}(F_l)$ , is the set  $\{x \in F_l \mid F_l - \{x\} \in \bigcup_{i=1}^{l-1} \bar{F}_i\}$ .

**Proposition 2.5** [Björner, Wachs [2]] Given an ordering  $F_1, \dots, F_m$  of the facets of  $\Lambda$  and a map  $\mathcal{R} : \{F_1, \dots, F_m\} \rightarrow \Lambda$ , the following are equivalent:

1.  $F_1, \dots, F_m$  is a shelling and  $\mathcal{R}$  its restriction map,

2. 
$$\left\{ \begin{array}{l} \Lambda = \bigsqcup_{i=1}^m [\mathcal{R}(F_i), F_i], \text{ and} \\ \mathcal{R}(F_i) \subseteq F_j \text{ implies } i \leq j, \text{ for all } i, j. \end{array} \right.$$

In Section 3, we will also determine a basis for the unique nontrivial cohomology group of  $\Lambda(H)$ . The cohomology groups of  $\Lambda(H)$  are defined in the usual manner. Please see Munkres [11] for more information on the computation of cohomology groups.

In Section 4, we will study the cyclic coloring complex of a complete  $k$ -uniform hypergraph. To define the cyclic coloring complex, we first must define an equivalence relation on the elements of  $\pm\Lambda_r(H)$ :

Let  $\sigma \in S_{r+2}$  be the  $(r+2)$ -cycle  $(1, 2, \dots, r+2)$ . Define  $\Delta_r(H) = \pm \Lambda_r(H) / \sim$ , where  $\sim$  is defined by  $(B_1, \dots, B_{r+2}) \sim (-1)^{r+1} (B_{\sigma(1)}, \dots, B_{\sigma(r+2)})$ . Let  $[B_1, \dots, B_{r+2}]$  denote the equivalence class containing  $(B_1, \dots, B_{r+2})$ . We will represent each equivalence class of  $\Delta_r(H)$  by the unique representative that has  $1 \in B_1$ . Let

$$\partial_r([B_1, \dots, B_{r+2}]) := \sum_{i=1}^{r+1} (-1)^{i+1} [B_1, \dots, B_i \cup B_{i+1}, \dots, B_{r+2}] + (-1)^{r+3} [B_1 \cup B_{r+2}, B_2, \dots, B_{r+1}].$$

It is straightforward to check that  $\partial$  is well-defined on equivalence classes.

**Definition 2.8.** *The cyclic coloring complex of  $H$ ,  $\Delta(H)$ , is the sequence*

$$\cdots \rightarrow C_r \xrightarrow{\partial_r} C_{r-1} \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} 0$$

where  $C_r$  is the vector space over a field of characteristic zero.

Notice that  $\partial_{r-1} \circ \partial_r = 0$ , so then:

**Definition 2.9.** *The  $r^{\text{th}}$  homology group of  $\Delta(H)$  is  $HC_r(\Delta(H)) = \ker(\partial_r) / \text{im}(\partial_{r+1})$ .*

As mentioned in Crown [5], the motivation for the definition of the cyclic coloring complex comes from cyclic homology. See Loday [9] for more information on cyclic homology.

In a couple of our arguments, we will consider the homology of the quotient of two cyclic coloring complexes, so we will define this quotient now:

Consider the cyclic coloring complex of a hypergraph  $H$ ,  $\Delta(H)$ , and consider a subcomplex,  $\Delta(I)$ , of  $\Delta(H)$ , where  $I$  is a subhypergraph of  $H$ . Then  $\Delta_r(H) / \Delta_r(I)$  will consist of the partitions  $[B_1, \dots, B_{r+2}]$  of  $\Delta(H)$  where none of the  $B_i$  contain a hyperedge of  $I$ . Thus, we obtain the sequence of complexes:

$$\Delta(I) \hookrightarrow \Delta(H) \xrightarrow{j} \Delta(H) / \Delta(I)$$

where  $i$  is the inclusion map and  $j$  is the quotient map. From the homology of the pair,  $(\Delta(H), \Delta(I))$ , this then induces the long exact sequence:

$$\cdots \rightarrow HC_r(\Delta(I)) \xrightarrow{i_*} HC_r(\Delta(H)) \xrightarrow{j_*} HC_r(\Delta(H) / \Delta(I)) \xrightarrow{\partial_*} HC_{r-1}(\Delta(I)) \rightarrow \cdots$$



where  $i_*$  is the map induced by the inclusion  $\Delta(I) \hookrightarrow \Delta(H)$ ,  $j_*$  is the map induced by the quotient map  $j$ , and  $\partial_*$  is the map induced by the boundary map  $\partial$ .

One of our results will relate the rank of the unique nontrivial homology group of  $\Lambda(H)$  to the chromatic polynomial of  $H$ ,  $\chi_H(\lambda)$ . So we include the definition of  $\chi_H(\lambda)$  here:

**Definition 2.10.** *A proper  $\lambda$ -coloring of  $H$  is a function  $f : V \rightarrow \{1, \dots, \lambda\}$  such that for each hyperedge,  $e$ , of  $H$  there exist at least two vertices  $v_1$  and  $v_2$  in  $e$  such that  $f(v_1) \neq f(v_2)$ . The chromatic polynomial of  $H$ , denoted  $\chi_H(\lambda)$ , is the polynomial whose value at  $\lambda$  gives the number of proper  $\lambda$ -colorings of  $H$ .*

### 3 The Coloring Complex

Suppose  $k \geq 3$ . In this section, we will show that for  $k > n/2$ , the coloring complex of a complete  $k$ -uniform hypergraph is shellable, and we will give a formula for the rank of the unique nontrivial homology group of the coloring complex in terms of the chromatic polynomial of the associated hypergraph.

**Theorem 3.1.** *Let  $H$  be the complete  $k$ -uniform hypergraph on  $n$  vertices and let  $k > n/2$ . Then  $\Lambda(H)$  is shellable of dimension  $n - k - 1$ .*

*Proof.* To show that  $\Lambda(H)$  is shellable, we will give an ordering of the the facets of  $\Lambda(H)$  and will show that there is a unique minimal new face introduced at the  $i^{th}$  step. This will allow us to deduce that  $\Lambda(H)$  is a disjoint union of Boolean intervals. From here it will follow from Proposition 2.5 in Björner and Wachs [2] that  $\Lambda(H)$  is shellable.

Notice that the facets of  $\Lambda(H)$  are ordered set partitions,  $F = (B_1, \dots, B_{n-k+1})$  of  $[n]$  where exactly one block has size  $k$  and the other blocks are singleton blocks. Let  $h(F)$  be the index of the block of size  $k$  and let  $w(F) = w_1 w_2 \dots w_{n-k}$  be the permutation of the elements of  $[n] \setminus B_{h(F)}$  obtained by listing these elements in the order that they appear in  $F$ . Let  $G = (B'_1, \dots, B'_{n-k+1})$  be a facet of  $\Lambda(H)$ , and consider the following ordering of the facets:

- If  $h(F) < h(G)$ , then  $F$  appears before  $G$ .

- If  $h(F) = h(G)$  and  $B_{h(F)}$  appears before  $B'_{h(G)}$  in the lexicographic order on  $k$ -sets, then  $F$  appears before  $G$ .
- If  $h(F) = h(G)$  and  $B_{h(F)} = B'_{h(G)}$ , then  $F$  appears before  $G$  if and only if  $w(F)$  appears before  $w(G)$  in the lexicographic order on permutations.

We now define a map  $\mathcal{R}$  from the set of facets to  $\Lambda(H)$ . Let the descent set of  $w(F)$  be  $\text{des}(w(F)) = \{d_1, \dots, d_l\}$ . If  $A_F = \{i \mid i < h(F) \text{ and } d_i \in \text{des}(w(F))\}$  is a nonempty set, then let  $t = \max\{i \mid i \in A_F\}$ . If  $w_{h(F)} > \max\{B_{h(F)}\}$ , then

$$\mathcal{R}(F) = (w_1 \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_{d_t} \cup \dots \cup w_{h(F)-1}, B_{h(F)} \cup w_{h(F)} \cup \dots \cup w_{d_{t+1}}, \dots, w_{d_l} \cup \dots \cup w_{n-k}).$$

Otherwise,

$$\mathcal{R}(F) = (w_1 \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_{d_t} \cup \dots \cup w_{h(F)-1}, B_{h(F)}, w_{h(F)} \cup \dots \cup w_{d_{t+1}}, \dots, w_{d_l} \cup \dots \cup w_{n-k}).$$

On the other hand, suppose  $A_F = \emptyset$ . If  $w_{h(F)} > \max\{B_{h(F)}\}$ , then

$$\mathcal{R}(F) = (w_1 \cup \dots \cup w_{h(F)-1}, B_{h(F)} \cup w_{h(F)} \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_{d_l} \cup \dots \cup w_{n-k}).$$

Otherwise,

$$\mathcal{R}(F) = (w_1 \cup \dots \cup w_{h(F)-1}, B_{h(F)}, w_{h(F)} \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_{d_l} \cup \dots \cup w_{n-k}).$$

Let  $F_1, \dots, F_m$  be the ordering of the facets of  $\Lambda(H)$  under the above ordering. We first show that  $\mathcal{R}(F_i)$  is a new face introduced at the  $i^{\text{th}}$  step. Notice that for a facet  $F_i = (w_1, \dots, w_{h(F_i)-1}, B_{h(F_i)}, w_{h(F_i)}, \dots, w_{n-k})$  that it follows from the ordering that for  $s \neq h(F_i) - 1$ , the face

$$(w_1, \dots, w_s \cup w_{s+1}, \dots, w_{h(F_i)-1}, B_{h(F_i)}, w_{h(F_i)}, \dots, w_{n-k})$$

is in  $\bigcup_{j=1}^{i-1} \bar{F}_j$  if and only if  $w_s > w_{s+1}$ . Since  $w_1 < \dots < w_{d_1}$ , it follows that the face

$$(w_1 \cup \dots \cup w_{d_1}, w_{d_1+1}, \dots, w_{h(F_i)-1}, B_{h(F_i)}, w_{h(F_i)}, \dots, w_{n-k})$$

is not in  $\bigcup_{j=1}^{i-1} \bar{F}_j$ . It then follows similarly that for  $A_{F_i} \neq \emptyset$

$$(w_1 \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_t \cup \dots \cup w_{h(F_i)-1}, B_{h(F_i)}, w_{h(F_i)} \cup \dots \cup w_{t+1}, \dots, w_{d_l} \cup \dots \cup w_{n-k})$$

is not in  $\bigcup_{j=1}^{i-1} \bar{F}_j$ . Further, notice that if  $w_{h(F_i)} < \max\{B_{h(F_i)}\}$ , then the face

$$(w_1, \dots, w_{h(F_i)-1}, B_{h(F_i)} \cup w_{h(F_i)}, w_{h(F_i)+1}, \dots, w_{n-k})$$

is in  $\bigcup_{j=1}^{i-1} \bar{F}_j$  since it is also a face of the facet

$$(w_1, \dots, w_{h(F_i)-1}, (B_{h(F_i)} - \{\max\{B_{h(F_i)}\}\}) \cup w_{h(F_i)}, \max\{B_{h(F_i)}\}, w_{h(F_i)+1}, \dots, w_{n-k}).$$

So if  $w_{h(F_i)} < \max\{B_{h(F_i)}\}$ , then

$$\mathcal{R}(F_i) = (w_1 \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_t \cup \dots \cup w_{h(F_i)-1}, B_{h(F_i)}, w_{h(F_i)} \cup w_{t+1}, \dots, w_{d_l} \cup \dots \cup w_{n-k})$$

is a new face. Otherwise,

$$\mathcal{R}(F_i) = (w_1 \cup \dots \cup w_{d_1}, w_{d_1+1} \cup \dots \cup w_{d_2}, \dots, w_t \cup \dots \cup w_{h(F_i)-1}, B_{h(F_i)} \cup w_{h(F_i)} \cup \dots \cup w_{t+1}, \dots, w_{d_l} \cup \dots \cup w_{n-k})$$

is a new face of the complex. We can similarly argue that if  $A_{F_i} = \emptyset$ ,  $\mathcal{R}(F_i)$  is a new face of the complex.

Notice that the minimality of  $\mathcal{R}(F_i)$  follows from the above observations as well as by noting that the face

$$(w_1, \dots, w_{h(F_i)-2}, w_{h(F_i)-1} \cup B_{h(F_i)}, w_{h(F_i)}, \dots, w_{n-k})$$

is a face of the facet

$$(w_1, \dots, w_{h(F_i)-2}, B_{h(F_i)}, w_{h(F_i)-1}, w_{h(F_i)}, \dots, w_{n-k}).$$

Since  $\mathcal{R}(F_i)$  is the unique minimal new face introduced at the  $i$ -th step, by induction, we have:

$$\Lambda(H) = \bigsqcup_{i=1}^m [\mathcal{R}(F_i), F_i].$$

From the above argument, it follows that  $\mathcal{R}(F_i) \subseteq F_j$  implies  $i \leq j$  for all  $i, j$ . Then by Proposition 2.5 in Björner and Wachs [2], the ordering  $F_1, \dots, F_m$  is a shelling order and  $\mathcal{R}$  is its restriction map, and  $\Lambda(H)$  is shellable.  $\square$

As noted in Björner [1], shellable complexes are Cohen-Macaulay. Thus,  $\Lambda(H)$  has a unique nontrivial homology group. By Theorem 4.1 of Long and Rundell [10], we obtain the following corollary:

**Corollary 3.2.** *If  $H$  is the complete,  $k$ -uniform hypergraph on  $n$  vertices, then the homology of  $\Lambda(H)$  is nonzero only in dimension  $n - k - 1$ , and the dimension of  $H_{n-k-1}(\Lambda(H))$  equals the sum of the absolute values of the coefficients of  $\chi_H(\lambda)$  minus one. Moreover, the dimension of the  $j^{\text{th}}$  Hodge piece in the Hodge decomposition of  $\Lambda(H)$  is*

$$\dim(H_{n-k-1}^{(j)}(\Lambda(H))) = (-1)^{n-k} [\lambda^j](\chi_H(-\lambda) - (-\lambda)^n).$$

From Theorem 3.1, we may obtain a basis for the reduced cohomology group  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$ . Notice that  $\Lambda(H)$  is a shellable pure complex, and let  $\Gamma$  be the set of homology facets of the shelling from the proof of Theorem 3.1, i.e. the set of facets  $F$  of  $\Lambda(H)$  such that  $\mathcal{R}(F) = F$ . By the definition of  $\mathcal{R}$ ,  $\Gamma$  is the set of facets  $F$  of  $\Lambda(H)$  such that  $\max\{B_{h(F)}\} > w_{h(F)}$  and the permutation  $w(F) = w_1 \dots w_{n-k}$  satisfies  $w_1 > \dots > w_{h(F)-1}$  and  $w_{h(F)} > \dots > w_{n-k}$ .

Following the notation in Björner and Wachs [2], for each  $F \in \Gamma$ , let  $\sigma^F$  denote a  $(n - k - 1)$ -cochain defined by

$$\sigma^F(G) = \begin{cases} 1 & \text{if } G = F \\ 0 & \text{if } G \neq F \end{cases}$$

for  $G \in \Lambda_{n-k-1}(H)$ . The fact that  $F$  is a facet of  $\Lambda(H)$  implies that  $\sigma^F$  is a cocycle and therefore determines a cohomology class  $[\sigma^F]$  in  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$ . Theorem 4.3 in Björner and Wachs [2] then gives the following corollary to Theorem 3.1:

**Corollary 3.3.** *Let  $\Gamma$  be the set of facets  $F$  of  $\Lambda(H)$  such that  $\max\{B_{h(F)}\} > w_{h(F)}$  and the permutation  $w(F) = w_1 \dots w_{n-k}$  satisfies  $w_1 > \dots > w_{h(F)-1}$  and  $w_{h(F)} > \dots > w_{n-k}$ . Then the classes  $[\sigma^F]$ , for  $F \in \Gamma$ , are a basis of  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$ .*

Notice that the condition that  $k > n/2$  is necessary in Theorem 3.1. If  $n/2 \geq k > 2$ , the complete  $k$ -uniform hypergraph contains a pair of disjoint edges. By Proposition 7 of Breuer, et al. [3],  $\Lambda(H)$  is not Cohen-Macaulay and hence not shellable.

From the shelling order and the argument given in the proof of Theorem 3.1 we also have the following result and corollaries:

**Theorem 3.4.** *Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices, and let  $v$  be a vertex of  $H$ . Suppose that the edge set of  $H$  consists of all possible hyperedges of size  $k$  containing the vertex  $v$ . Then  $\Lambda(H)$  is shellable.*

**Corollary 3.5.** *Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices, and let  $v$  be a vertex of  $H$ . Suppose that  $H$  consists of all possible hyperedges of size  $k$  containing the vertex  $v$ . Then the homology of  $\Lambda(H)$  is nonzero only in dimension  $n - k - 1$ , and the dimension of  $H_{n-k-1}(\Lambda(H))$  equals the sum of the absolute values of the coefficients of  $\chi_H(\lambda)$  minus one. Moreover, the dimension of the  $j^{\text{th}}$  Hodge piece in the Hodge decomposition of  $\Lambda(H)$  is*

$$\dim(H_{n-k-1}^{(j)}(\Lambda(H))) = (-1)^{n-k} [\lambda^j] (\chi_H(-\lambda) - (-\lambda)^n).$$

**Corollary 3.6.** *Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices, and let  $v$  be a vertex of  $H$ . Suppose that the edge set of  $H$  consists of all possible hyperedges of size  $k$  containing the vertex  $v$ . Let  $\Gamma$  be the set of facets  $F$  of  $\Lambda(H)$  where  $v \in B_{h(F)}$ ,  $\max\{B_{h(F)}\} > w_{h(F)}$ , and the permutation  $w(F)$  satisfies  $w_1 > \dots > w_{h(F)-1}$  and  $w_{h(F)} > \dots > w_{n-k}$ . Then the classes  $[\sigma^F]$ , for  $F \in \Gamma$ , are a basis of  $\tilde{H}^{n-k-1}(\Lambda(H), \mathbb{Z})$ .*

## 4 The Cyclic Coloring Complex

In this section, we discuss the homology of the cyclic coloring complex of a complete  $k$ -uniform hypergraph. In the proof of Theorem 4.1, we will use a spectral sequence argument similar to the proof of Theorem 3.2 in Crown [5]. Thus, before presenting Theorem 4.1 and its proof, we provide a summary of some of the results of Crown [5] that we will need in the proof of Theorem 4.1.

**Theorem 3.2** [Crown [5]] *Let  $\Delta(E_n)$  be the cyclic coloring complex of the complete graph with looped edges at each vertex. The dimension of the  $r$ -th homology group of  $\Delta(E_n)$ ,  $HC_r(\Delta(E_n))$ , is  $\binom{n-1}{r+1}$ .*

The proof of Theorem 3.2 begins by considering the function

$$F([B_1, \dots, B_{r+2}]) = |B_1|$$

where  $B_1$  is the block containing the vertex 1. Let  $\Delta_r^m$  denote the elements of  $\Delta_r$  such that

$$F([B_1, \dots, B_{r+2}]) = m,$$

and let  $\Delta(E_n^m)$  denote the complex formed by the chains in  $\Delta_r^m$ ,  $-1 \leq r \leq n-2$ . Notice that  $f$  gives a grading of each  $\Delta_r(E_n)$ . Define  $\Delta_r^{(m)}(E_n) = \bigcup_{m \leq i \leq n} \Delta_r(E_n^i)$ . When the boundary map  $\partial$  is applied to an element of  $\Delta_r(E_n^i)$  the result is a signed sum of elements where  $|B_1| = i$  or  $|B_1| = i+1$ . Therefore,  $\partial(\Delta_r^{(m)}(E_n)) \subseteq \Delta_{r-1}^m(E_n)$ , and thus  $\partial$  respects the grading.

The proof of Theorem 3.2 then uses a spectral sequence argument to determine the dimensions of the homology groups of  $\Delta(E_n)$  and follows the construction and notation of Chow [4]. Let  $C_{r,m}$  be the vector space with basis  $\Delta_r^{(m)}(E_n)$ . Then  $E_{r,m}^0 = C_{r,m}/C_{r,m+1}$ , and in particular,  $E_{r,m}^0$  is the vector space with basis  $\Delta_r(E_n^m)$  and  $C_r \cong \bigoplus_{m=1}^n E_{r,m}^0$ . Further,  $\partial$  induces a map

$$\partial^0 : \bigoplus_{m=1}^n E_{r,m}^0 \rightarrow E_{r-1,m}^0$$

where  $\partial^0(E_{r,m}^0) \subseteq E_{r-1,m}^0$  for all values of  $r, m$ . One can then define:

$$E_{r,m}^1 = HC_r(E_{r,m}^0) = \frac{\ker \partial^0 : E_{r,m}^0 \rightarrow E_{r-1,m}^0}{\text{im } \partial^0 : E_{r+1,m}^0 \rightarrow E_{r,m}^0}.$$

Further,  $\partial$  induces a map:

$$\partial^1 : E_{r,m}^1 \rightarrow E_{r-1,m+1}^1$$

and we can define

$$E_{r,m}^2 = HC_r(E_{r,m}^1) = \frac{\ker \partial^1 : E_{r,m}^1 \rightarrow E_{r-1,m+1}^1}{\text{im } \partial^1 : E_{r+1,m-1}^1 \rightarrow E_{r,m}^1}.$$

In the proof of Theorem 3.2,  $E_{r,m}^2 = 0$  for all values of  $r$  and  $m$  which implies that  $HC_r(\Delta(E_n)) = \bigoplus_{-1 \leq r \leq n-2} E_{r,m}^1$ .

To compute the dimension of  $HC_r(\Delta(H))$ , Crown notes that the elements of  $\Delta^m(H)$  can be partitioned into subcomplexes determined by the elements of  $B_1$ . Each of these subcomplexes has the homology of a Boolean algebra of  $n-m$  elements. A homology representative of the unique nontrivial homology group of the subcomplex is then  $\sum_{\sigma \in S_{n-m}} \text{sgn}(\sigma) [B_1, a_{\sigma(1)}, \dots, a_{\sigma(n-m)}]$ , where  $\{a_1, \dots, a_{n-m}\}$  are the elements of the set

$[n] \setminus B_1$ . There are  $\binom{n-1}{m-1}$  subcomplexes of  $\Delta^m(E_n)$ , and since  $E_{r,m} = 0$  for all values of  $r$  and  $m$ , it follows that the dimension of  $HC_r(\Delta(E_n))$  is given by  $\binom{n-1}{r+1}$ .

**Theorem 4.1.** *Let  $H$  be the  $k$ -uniform hypergraph on  $n$  vertices with edge set consisting of all possible hyperedges of size  $k$  containing the vertex 1. Then the dimension of  $HC_r(\Delta(H))$  is nonzero for  $n - k - 1 \geq r \geq -1$  and is given by*

$$\dim(HC_r(\Delta(H))) = \binom{n-1}{r+1}.$$

*Proof.* Consider the function

$$f([B_1, \dots, B_{r+2}]) = |B_1|$$

where  $B_1$  is the block containing 1. Let  $\Delta_r^m(H)$  denote the elements in  $\Delta_r(H)$  where

$$f([B_1, \dots, B_{r+2}]) = m,$$

and let  $\Delta^m(H)$  denote the complex formed by the elements in  $\Delta_i^m(H)$ ,  $-1 \leq i \leq n-2$ . Notice that  $f$  gives a grading of each  $\Delta_r(H)$  and that the boundary map  $\partial$  respects the grading. We will use a spectral sequence argument to determine the dimensions of the homology groups of  $\Delta(H)$ .

Notice that  $\Delta^m(E_n) = \Delta^m(H)$  for  $n - k + 1 \leq m \leq n$ . As in the proof of Theorem 3.2 in Crown [5], we can see that the elements of  $\Delta^m(H)$  can be partitioned into subcomplexes determined by the elements of  $B_1$ , and each subcomplex has the homology of a Boolean algebra of  $n - m$  elements. In particular, each subcomplex has a unique nontrivial homology group of rank one with homology representative given by

$$\sum_{\sigma \in S_{n-m}} \text{sgn}(\sigma) [B_1, a_{\sigma(1)}, \dots, a_{\sigma(n-m)}], \text{ where } \{a_1, \dots, a_{n-m}\} \text{ are the elements of the set } [n] \setminus B_1.$$

Consider  $\partial^1(\sum_{\sigma \in S_{n-m}} \text{sgn}(\sigma) [B_1, a_{\sigma(1)}, \dots, a_{\sigma(n-m)}])$ . Recall that  $\partial^1$  is the map obtained by taking those terms of the boundary map,  $\partial$ , in which the size of  $B_1$  increased by one. So,

$$\partial^1([B_1, a_1, \dots, a_{n-m}]) = [B_1 \cup a_1, a_2, \dots, a_{n-m}] + (-1)^{n-m} [B_1 \cup a_{n-m}, a_1, \dots, a_{n-m-1}].$$

Let  $\pi$  be a permutation in  $S_{n-m}$  and let  $\pi(1) = i$ . There exists a unique permutation  $\tau$  in  $S_{n-m}$  for which

$$\tau(n-m) = i \text{ and } \tau(1) = \pi(2), \dots, \tau(n-m-1) = \pi(n-m).$$

Notice that when we apply  $\partial^1$  to  $\text{sgn}(\pi)[B_1, a_{\pi(1)}, \dots, a_{\pi(n-m)}]$  and  $\text{sgn}(\tau)[B_1, a_{\tau(1)}, \dots, a_{\tau(n-m)}]$ , each resulting sum will have the term  $[B_1 \cup a_i, a_{\pi(2)}, \dots, a_{\pi(n-m)}]$ . It remains to show that the coefficients of these terms cancel. The coefficient of  $[B_1 \cup a_i, a_{\pi(2)}, \dots, a_{\pi(n-m)}]$  in  $\partial^1(\text{sgn}(\pi)[B_1, a_{\pi(1)}, \dots, a_{\pi(n-m)}])$  is  $\text{sgn}(\pi)$ . Note that  $\text{sgn}(\tau) = (-1)^{n-m-1} \text{sgn}(\pi)$ , and thus the coefficient of  $[B_1 \cup a_i, a_{\pi(2)}, \dots, a_{\pi(n-m)}]$  in  $\partial^1(\text{sgn}(\tau)[B_1, a_{\tau(1)}, \dots, a_{\tau(n-m)}])$  is  $(-1)^{n-m-1} \text{sgn}(\pi) * (-1)^{n-m} = -\text{sgn}(\pi)$ . Thus,

$$\partial^1\left(\sum_{\sigma \in S_{n-m}} \text{sgn}(\sigma)[B_1, a_{\sigma(1)}, \dots, a_{\sigma(n-m)}]\right) = 0.$$

The same argument holds for  $\partial^2, \partial^3, \dots$ , and thus the spectral sequence collapses.

It then follows that to determine the dimension of the homology group  $HC_r(\Delta(H))$ , we must relate  $r$  to  $m$  and determine the number of subcomplexes of  $\Delta^m(H)$ . Notice that  $r = (n - 2) - (m - 1)$  and that the number of subcomplexes of  $\Delta^m(H)$  is given by the number of ways of forming a subset of size  $m - 1$  from a set of size  $n - 1$ . Thus,

$$\dim(HC_r(\Delta(H))) = \binom{n-1}{r+1}. \quad \square$$

Notice that the proof of Theorem 4.1 shows that the homology representatives of the  $r^{\text{th}}$  homology group of  $\Delta(H)$  are indexed by the subsets of size  $n - r - 2$  of  $\{2, \dots, n\}$ . Namely, for each subset,  $A$ , of size  $n - r - 2$  of  $\{2, \dots, n\}$ , we obtain one homology representative of  $HC_r(\Delta(H))$ ,  $\sum_{\sigma \in S_{r+1}} \text{sgn}(\sigma)[A \cup \{1\}, a_{\sigma(1)}, \dots, a_{\sigma(r+1)}]$ , where  $\{a_1, \dots, a_{r+1}\}$  is the complement of  $A$  in  $\{2, \dots, n\}$ .

We can define an action of  $S_{r+2}$  on  $\Delta_r$ , and this action then makes  $C_r$  into an  $S_{r+2}$ -module. Namely, if  $\sigma \in S_{r+2}$ , then  $\sigma \cdot (B_1, \dots, B_{r+2}) = (B_{\sigma^{-1}(1)}, \dots, B_{\sigma^{-1}(r+2)})$ . It is possible to describe the  $S_n$ -module structure of the  $r$ -th homology group of  $\Delta(E_n)$ , and hence, the  $S_n$ -module structure of the  $r$ -th homology group of  $\Delta(H)$ , where  $H$  is a complete  $k$ -uniform hypergraph.

**Theorem 4.2.** *The  $S_n$ -module structure of  $HC_r(\Delta(E_n))$  is  $S^\lambda$  where  $\lambda = (n - r - 1, 1^{r+1})$ . Moreover, this is the  $S_n$ -module structure of the multilinear part of  $HC_r(\mathbb{C}[x_1, \dots, x_n])$ .*

*Proof.* For each subset  $A$ , of size  $n - r - 2$  of  $\{2, \dots, n\}$ , we obtain one homology representative of  $HC_r(\Delta(E_n))$ , namely  $\sum_{\sigma \in S_{r+1}} \text{sgn}(\sigma)[A \cup \{1\}, a_{\sigma(1)}, \dots, a_{\sigma(r+1)}]$ . Let  $B_1 = A \cup \{1\}$ .



Since  $1 \in B_1$ , consider first the  $S_{n-1}$ -module structure of  $HC_r(\Delta(E_n))$ , where  $S_{n-1}$  is acting on  $\{2, \dots, n\}$ . The homology representative  $\sum_{\sigma \in S_{r+1}} \text{sgn}(\sigma)[B_1, a_{\sigma(1)}, \dots, a_{\sigma(r+1)}]$ , is invariant under permutations of the elements of the set  $B_1 \setminus \{1\}$ , and if the elements of the set  $\{a_1, \dots, a_{r+1}\}$  are permuted by  $\tau$ , then the homology representative is mapped to

$$\text{sgn}(\tau) \sum_{\sigma \in S_{r+1}} \text{sgn}(\sigma)[B_1, a_{\sigma(1)}, \dots, a_{\sigma(r+1)}].$$

Thus,

$$HC_r(\Delta(E_n)) \downarrow_{S_{n-1}} = (S^{(n-r-2)} \otimes S^{(1^{r+1})}).$$

By the Littlewood-Richardson Rule, we then have:

$$HC_r(\Delta(E_n)) \downarrow_{S_{n-1}} = S^{(n-r-1, 1^r)} \oplus S^{(n-r-2, 1^{r+1})}.$$

If  $\lambda$  is a partition of  $n$ , then  $S^\lambda \downarrow_{S_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-}$  where  $\lambda^-$  is a partition obtained from  $\lambda$  by removing one block. We similarly use the notation  $\lambda^+$  for a partition obtained from  $\lambda$  by adding one block.

We wish to show that  $V = S^{(n-r-1, 1^{r+1})}$  is the only representation such that  $V \downarrow_{S_{n-1}} = S^{(n-r-1, 1^r)} \oplus S^{(n-r-2, 1^{r+1})}$ , for  $n \geq 5$ . (We treat the cases  $n = 3$  and  $n = 4$  separately below.) Suppose  $V = \sum_{\mu} c_{\mu} S^{\mu}$ . Notice that for  $\lambda_1 = (n-r-1, 1^r)$ , the possibilities for  $\lambda_1^+$  are  $(n-r, 1^r)$ ,  $(n-r-1, 1^{r+1})$ , and  $(n-r-1, 2, 1^{r-1})$ . For  $\lambda_2 = (n-r-2, 1^{r+1})$ , the possibilities for  $\lambda_2^+$  are  $(n-r-1, 1^{r+1})$ ,  $(n-r-2, 1^{r+2})$ , and  $(n-r-2, 2, 1^r)$ . So,

$$V = c_1 S^{(n-r, 1^r)} + c_2 S^{(n-r-1, 1^{r+1})} + c_3 S^{(n-r-1, 2, 1^{r-1})} + c_4 S^{(n-r-2, 1^{r+2})} + c_5 S^{(n-r-2, 2, 1^r)}.$$

Consider the partitions  $(n-r-1, 2, 1^{r-1})$  and  $(n-r-2, 2, 1^r)$ . In the first case, notice that one of the partitions that is obtained by removing a block is the partition  $(n-r-1, 2, 1^{r-2})$ . When we restrict  $V$ , the only other possible partition that could contribute a term to cancel the term corresponding to  $(n-r-1, 2, 1^{r-2})$  is the partition  $(n-r-2, 2, 1^r)$ . However, if we remove a block from the partition  $(n-r-2, 2, 1^r)$ , we obtain the partition  $(n-r-3, 2, 1^r)$ ,  $(n-r-2, 1^{r+1})$ , or  $(n-r-2, 2, 1^{r-1})$ . Since none of these partitions are equal to  $(n-r-1, 2, 1^{r-2})$ ,  $c_3 = 0$ . By a similar argument,  $c_5 = 0$ .

Now consider the partitions  $(n-r, 1^r)$  and  $(n-r-2, 1^{r+2})$ . For the former, if we remove a block from the partition, we obtain either the partition  $(n-r, 1^{r-1})$  or the

partition  $(n - r - 1, 1^r)$ . When we restrict  $V$ , the only other possible partition that could contribute terms to cancel the terms corresponding to  $(n - r, 1^{r-1})$  and  $(n - r - 1, 1^r)$  is the partition  $(n - r - 2, 1^{r+2})$ . However, notice that when we remove a block from the partition  $(n - r - 2, 1^{r+2})$  we obtain either the partition  $(n - r - 3, 1^{r+1})$  or the partition  $(n - r - 2, 1^r)$ . Since neither of these partitions is equal to  $(n - r - 1, 1^r)$  or  $(n - r, 1^{r-1})$ ,  $c_1 = c_4 = 0$ . Thus, for  $n \geq 5$ , the restriction  $S^{(n-r-1, 1^{r+1})} \downarrow_{S_{n-1}}$  equals  $S^{\lambda_1} \oplus S^{\lambda_2}$ .

Suppose that  $n = 3$ . For  $r = 1$  and  $r = -1$ , the above argument holds. In particular, the argument shows that the  $S_n$ -module structure of  $HC_1(\Delta(E_3))$  is  $S^{(1^3)}$ , and the  $S_n$ -module structure of  $HC_{-1}(\Delta(E_3))$  is  $S^{(3)}$ . Suppose then that  $r = 0$ . The elements  $[12, 3]$  and  $[13, 2]$  are a set of homology representatives for  $HC_0(\Delta(E_3))$ . It is straightforward to see that the character on the conjugacy class indexed by the identity element equals 2, the character on the conjugacy class indexed by cycle type  $(2, 1)$  is 0, and the character on the conjugacy class indexed by cycle type  $(3)$  is  $-1$ . Thus, the  $S_n$ -module structure of  $HC_0(\Delta(E_n))$  is  $S^{(2,1)}$ .

Suppose  $n = 4$ . The cases  $r = 2$  and  $r = -1$  can be proven using the restriction argument above. For the case  $r = 0$ ,  $[123, 4]$ ,  $[124, 3]$ , and  $[134, 2]$  are a set of homology representatives for  $HC_0(\Delta(E_4))$ . It is straightforward to see that the character on the conjugacy class indexed by the identity element is 3, the character on the conjugacy class indexed by cycle type  $(2, 1, 1)$  is 1, the character on the conjugacy class indexed by cycle type  $(2, 2)$  is  $-1$ , the character on the conjugacy class indexed by cycle type  $(3, 1)$  is zero, and the character on the conjugacy class indexed by cycle type  $(4)$  equals  $-1$ . Therefore, the  $S_4$ -module structure of  $HC_0(\Delta(E_4))$  is  $S^{(3,1)}$ . For the case  $r = 1$ , the argument is similar, and it can be seen that the  $S_4$ -module structure of  $HC_1(\Delta(E_4))$  is  $S^{(2,1,1)}$ .  $\square$

**Corollary 4.3.** *Let  $H$  be a hypergraph on  $n$  vertices, let  $v$  be a vertex of  $H$ , and let the hyperedges of  $H$  be all possible subsets of  $[n]$  of size  $k$  that contain vertex  $v$ . Then the  $S_n$ -module structure of  $HC_r(\Delta(H))$  is  $S^\lambda$  where  $\lambda = (n - r - 1, 1^{r+1})$ .*

Let  $H$  be the complete  $k$ -uniform hypergraph on  $n$  vertices. Notice that  $\Delta(H)$  is a subcomplex of  $\Delta(E_n)$ . Let  $\Delta(H)^C = \Delta(E_n)/\Delta(H)$ . Also, notice that if we apply the boundary map of  $\Delta(H)^C$  to a partition  $[B_1, \dots, B_{r+2}]$  and if one of the terms in the image

contains an edge of  $H$ , then this particular term is equal to zero. To compute the homology of  $\Delta(H)$ , we will first compute the homology of  $\Delta(H)^C$ .

**Theorem 4.4.** *Let  $H$  be the complete  $k$ -uniform hypergraph on  $n$  vertices.*

*For  $n - 2 \geq r > n - k$ ,*

$$\dim(HC_r(\Delta(H)^C)) = \binom{n-1}{r+1}$$

*and*

$$\dim(HC_{n-k}(\Delta(H)^C)) = \binom{n-1}{n-k-1} + \binom{n-1}{n-k+1}.$$

Before presenting the proof of Theorem 4.4, we will need the following definitions and lemma from Crown [5].

Let  $T_n$  be a tree on  $n$  vertices. Let the root of the tree be labeled 1, and label the other vertices  $2, \dots, n$  so that each parent node has a smaller vertex label than each of its children. Consider listing each edge of  $T_n$  by placing the smaller vertex first and order the edges in lexicographic order. Let  $\Delta(T_n^{(0,l)})$  be the complex formed by the elements  $[B_1, \dots, B_{r+2}]$  where none of the  $B_i$  contain one of the first  $l$  edges.

**Lemma 3.3** [Crown [5]] Given a tree on  $n$  vertices, for  $r \leq n - 3$ ,

$$\dim(\Delta(T_n^{(0,l)})) = \binom{n-(l+1)}{(r+2)-(l+1)}.$$

We now prove Theorem 4.4:

*Proof.* For the first part of the theorem, consider the following long exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{n-2}(\Delta(H)) & \rightarrow & HC_{n-2}(\Delta(E_n)) & \rightarrow & HC_{n-2}(\Delta(H)^C) \rightarrow \\ & & HC_{n-3}(\Delta(H)) & \rightarrow & HC_{n-3}(\Delta(E_n)) & \rightarrow & HC_{n-3}(\Delta(H)^C) \rightarrow \\ & & HC_{n-4}(\Delta(H)) & \rightarrow & HC_{n-4}(\Delta(E_n)) & \rightarrow & HC_{n-4}(\Delta(H)^C) \rightarrow \\ & & & & \vdots & & \end{array}$$

Notice that since  $HC_r(\Delta(H)) = 0$  for  $r > n - k - 1$ , by exactness,  $\dim HC_r(\Delta(H)^C) = \dim HC_r(\Delta(E_n)) = \binom{n-1}{r+1}$  for  $n - 2 \geq r > n - k$ .

We will now prove the second statement of the theorem. Let  $W_1, \dots, W_{\binom{n}{k}}$  be the subsets of  $\{1, \dots, n\}$  of size  $k$ , listed in lexicographic order. Let  $\Delta(E_n^{(0,W_i)})$  be the complex

formed by the chains  $[B_1, \dots, B_{r+2}]$  where for all  $i$ ,  $1 \leq i \leq l$ , the elements of  $W_i$  are not in the same block of the partition. Let  $\Delta(E_n^{(1, W_l)})$  be the complex formed by the chains  $[B_1, \dots, B_{r+2}]$  where for all  $i$ ,  $1 \leq i \leq l-1$ , the elements of  $W_i$  are not in the same block of the partition, but the elements of  $W_l$  are in the same  $B_j$ , for some  $j$ ,  $1 \leq j \leq r+2$ . Notice then that:

$$\Delta(E_n^{(0, W_{l-1})}) / \Delta(E_n^{(1, W_l)}) = \Delta(E_n^{(0, W_l)}).$$

Using this notation,  $\Delta(H)^C = \Delta(E_n^{(0, W_{\binom{n}{k}})})$ . We will compute the homology of  $\Delta(H)^C$  by sequentially computing the homology of  $\Delta(E_n^{(0, W_l)})$ .

Let  $H'$  be the  $k$ -uniform hypergraph on  $n$  vertices with edge set consisting of all possible hyperedges of size  $k$  containing the vertex 1. Let  $p = \binom{n-1}{k-1}$ . Since  $1 \in W_l$  for  $1 \leq l \leq p$ ,  $\Delta(H')^C = \Delta(E_n^{(0, W_p)})$ . We begin by using the following long exact sequence to calculate the homology of  $\Delta(H')^C$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{n-2}(\Delta(H')) & \xrightarrow{\alpha_{n-2}} & HC_{n-2}(\Delta(E_n)) & \rightarrow & HC_{n-2}(\Delta(H')^C) \rightarrow \\ & & HC_{n-3}(\Delta(H')) & \xrightarrow{\alpha_{n-3}} & HC_{n-3}(\Delta(E_n)) & \rightarrow & HC_{n-3}(\Delta(H')^C) \rightarrow \\ & & HC_{n-4}(\Delta(H')) & \xrightarrow{\alpha_{n-4}} & HC_{n-4}(\Delta(E_n)) & \rightarrow & HC_{n-4}(\Delta(H')^C) \rightarrow \\ & & & & \vdots & & \end{array}$$

By the same argument as in the first part of this proof, we can see that the dimension of  $HC_r(\Delta(H'))^C$  for  $n-2 \geq r \geq n-k+1$  is  $\binom{n-1}{r+1}$ . So consider  $r = n-k$ . We noted after the proof of Theorem 4.1 that the homology representatives of the  $r^{th}$  homology group of  $\Delta(H')$  are indexed by the subsets of size  $n-r-2$  of  $\{2, \dots, n\}$ . As noted in the proof of Theorem 3.2 in Crown [5], these are the same as the homology representatives of the  $r^{th}$  homology group of  $\Delta(E_n)$ . Thus  $\alpha_r$  is bijective for all  $r \leq n-k-1$ . By the above argument, it suffices to consider the exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{n-k}(\Delta(E_n)) & \rightarrow & HC_{n-k}(\Delta(H')^C) & \xrightarrow{\phi_{n-k}} & \\ HC_{n-k-1}(\Delta(H')) & \xrightarrow{\alpha_{n-k-1}} & HC_{n-k-1}(\Delta(E_n)) & \rightarrow & 0 & & \end{array}$$

Since  $\alpha_{n-k-1}$  is injective, the image of  $\phi_{n-k}$  is zero. Therefore, the dimension of  $HC_{n-k}(\Delta(H')^C)$  equals the dimension of the kernel of  $\phi_{n-k}$ . By exactness, the dimension of the kernel of

$\phi_{n-k}$  equals the dimension of  $HC_{n-k}(\Delta(E_n))$ . So the dimension of  $HC_{n-k}(\Delta(H')^C)$  is  $\binom{n-1}{n-k+1}$ .

Now we will sequentially compute  $HC_r(\Delta(H)^C)$ . Let  $l > p$ . Notice that  $HC_r(\Delta(E_n^{(1,W_l)}))$  is zero for  $r \geq n - k$ . So by exactness, for  $r > n - k$ ,

$$\dim(HC_r(\Delta(E_n^{(0,W_l)}))) = \dim(HC_r(\Delta(H')^C)) = \binom{n-1}{r+1}.$$

Further, we know  $HC_r(\Delta(H')^C) = 0$  for  $r \leq n - k - 1$ . It suffices then to consider the exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{n-k}(\Delta(H')^C) & \rightarrow & HC_{n-k}(\Delta(E_n^{(0,W_{p+1})})) & \rightarrow & \\ HC_{n-k-1}(\Delta(E_n^{(1,W_{p+1})})) & \xrightarrow{\alpha_{n-k-1}} & 0 & \rightarrow & HC_{n-k-1}(\Delta(E_n^{(0,W_{p+1})})) & \rightarrow & \\ HC_{n-k-2}(\Delta(E_n^{(1,W_{p+1})})) & \xrightarrow{\alpha_{n-k-2}} & 0 & \rightarrow & HC_{n-k-2}(\Delta(E_n^{(0,W_{p+1})})) & \rightarrow & \\ & & \vdots & & & & \end{array}$$

By exactness,  $HC_r(\Delta(E_n^{(0,W_{p+1})})) \cong HC_{r-1}(\Delta(E_n^{(1,W_{p+1})}))$  for  $n - k - 1 \geq r \geq 0$ . Consider the complex  $\Delta(E_n^{(1,W_{p+1})})$ . Notice that the homology of this complex is equal to the homology of the complex  $\Delta(T_{n-k+1}^{(0,1)})$ , where  $T_{n-k+1}$  is the tree on  $n - k + 1$  vertices, with root labeled 1, and edges  $(1, 2), (1, 3), \dots, (1, n - k + 1)$ , and where vertices 1 and 2 are not in the same block of a chain  $[B_1, \dots, B_{r+2}]$ . Let  $\{a_1, \dots, a_{n-k-1}\}$  be the elements of the complement of  $W_{p+1}$  in  $\{2, \dots, n\}$  listed in increasing order. The isomorphism is given by mapping 1 to 1,  $W_{p+1}$  to 2,  $a_1$  to 3,  $\dots$ ,  $a_{n-k-1}$  to  $n - k + 1$ . By Lemma 3.3 of Crown [5], the dimension of  $HC_{n-k-1}(\Delta(E_n^{(1,W_{p+1})}))$  is equal to  $\binom{(n-k+1)-(1+1)}{((n-k-1)+2)-(1+1)} = 1$ , and therefore,  $\dim(HC_{n-k}(\Delta(E_n^{(0,W_{p+1})}))) = 1 + \binom{n-1}{n-k+1}$ .

We claim that for each subsequent set  $W_l$  removed, the contribution to  $\dim HC_{n-k}(\Delta(H)^C)$  will be one. Notice that for all  $l$ ,  $p + 1 \leq l \leq \binom{n}{k}$ , the homology of the complex  $\Delta(E_n^{(1,W_l)})$  is equal to the homology of the complex  $\Delta(T_{n-k+1}^{(0,i)})$  for some  $i$ . By Lemma 3.3 of Crown, the dimension of  $HC_{n-k-1}(\Delta(E_n^{(0,W_l)}))$  is equal to  $\binom{(n-k+1)-(l+1)}{((n-k-1)+2)-(l+1)} = 1$ . By exactness, for each subsequent set  $W_l$  removed, the contribution to  $\dim(HC_{n-k}(\Delta(H)^C))$  will be one. Since there are  $\binom{n-1}{k}$  sets  $W_l$  of size  $k$  of  $[n - 1]$ ,  $\dim(HC_{n-k}(\Delta(H)^C)) = \binom{n-1}{k} + \binom{n-1}{n-k+1} = \binom{n-1}{n-k-1} + \binom{n-1}{n-k+1}$ .  $\square$

Notice that the elements of  $\Delta_r(H)^C$  are in bijection with the cyclic words  $[D_1, \dots, D_{r+2}]$  where  $D_i \in \mathbb{C}[x_1, \dots, x_n]/\{x_{i_1} \dots x_{i_k} \mid i_1 \dots i_k \text{ is a hyperedge of } H\}$  and  $[D_1, \dots, D_{r+2}]$  is an ordered partition of  $x_1 \dots x_n$  with  $x_1 \in D_1$ . It then follows that we have the following corollary:

**Corollary 4.5.** *For the complete  $k$ -uniform hypergraph on  $n$  vertices,  $H$ , the dimension of the multilinear part of the  $r^{\text{th}}$  cyclic homology group of  $\mathbb{C}[x_1, \dots, x_n]/\{x_{i_1} \dots x_{i_k} \mid i_1 \dots i_k \text{ is a hyperedge of } H\}$  is  $\binom{n-1}{r+1}$  for  $n-k \leq r \leq n-2$  and  $\binom{n-1}{n-k-1} + \binom{n-1}{n-k+1}$  for  $r = n-k$ .*

We will now determine the dimension the  $(n-k-1)^{\text{st}}$  homology group of  $\Delta(H)$  for a complete  $k$ -uniform hypergraph:

**Theorem 4.6.** *Let  $H$  be a complete  $k$ -uniform hypergraph. Then*

$$\dim(HC_{n-k-1}(\Delta(H))) = \binom{n}{n-k}.$$

*Proof.* Consider the following long exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{n-k}(\Delta(E_n)) & \rightarrow & HC_{n-k}(\Delta(H)^C) & \xrightarrow{\phi_{n-k}} & \\ HC_{n-k-1}(\Delta(H)) & \xrightarrow{\alpha_{n-k-1}} & HC_{n-k-1}(\Delta(E_n)) & \xrightarrow{\beta_{n-k-1}} & HC_{n-k-1}(\Delta(H)^C) & \rightarrow & \dots \end{array}$$

As noted in the proof of Theorem 4.4, each of the homology representatives of  $HC_r(\Delta(E_n))$  corresponds to a subset,  $A$ , of  $\{2, \dots, n\}$  of size  $n-r-2$ . Since the set  $\{1\} \cup A$  is an edge of  $H$ , each of the homology representatives of  $HC_{n-k-1}(\Delta(E_n))$  is mapped to zero by the map  $\beta_{n-k-1}$ . Therefore, the dimension of the kernel of  $\beta_{n-k-1}$  is  $\binom{n-1}{n-k}$ . By exactness and Theorem 3.2 of Crown [5], the dimension of the kernel of  $\phi_{n-k} = \binom{n-1}{n-k+1}$ . Thus,

$$\begin{aligned} \dim(HC_{n-k-1}(\Delta(H))) &= \left( \dim(HC_{n-k}(\Delta(H)^C)) - \binom{n-1}{n-k+1} \right) + \binom{n-1}{n-k} \\ &= \binom{n-1}{n-k-1} + \binom{n-1}{n-k} \\ &= \binom{n}{n-k} \end{aligned} \quad \square$$

When  $k = n-1$  and  $k = n-2$ , we have the results:

**Theorem 4.7.** *Let  $H$  be the complete  $(n-1)$ -uniform hypergraph on  $n$  vertices. Then*

$$\dim(HC_0(\Delta(H))) = \binom{n}{1} = n$$

and

$$\dim(HC_{-1}(\Delta(H))) = \binom{n}{0} = 1.$$

*Proof.* Note that there  $\binom{n}{1}$  elements in  $\Delta_0(H)$  and each of these elements is mapped to zero under  $\partial_0$ . Thus,  $\dim(HC_0(\Delta(H))) = n$ . Since  $\Delta_{-1}(H) = \{[12 \dots n]\}$ , it follows that  $\dim(HC_{-1}(\Delta(H))) = 1$ .  $\square$

**Theorem 4.8.** *Let  $H$  be the complete  $(n-2)$ -uniform hypergraph on  $n$  vertices. Then for  $-1 \leq r \leq 1$ ,*

$$\dim(HC_r(\Delta(H))) = \binom{n}{r+1}.$$

*Proof.* From Theorem 4.6, we know that  $\dim(HC_1(\Delta(H))) = \binom{n}{2}$ . The set  $\Delta_1(H)$  consists of all ordered partitions  $[B_1, B_2, B_3]$  where  $\{1\} \in B_1$  and where one of the  $B_i$  is a hyperedge of  $H$ . It follows then that there are  $2\binom{n-1}{n-3} + 2\binom{n-1}{n-2} = 2\binom{n}{n-2}$  elements in  $\Delta_1(H)$ . Thus the dimension of the image of  $\partial_1$  is  $\binom{n}{2}$ . The dimension of the kernel of  $\partial_0$  equals the cardinality of  $\Delta_0(H)$ . Since  $\Delta_0(H)$  consists of all ordered partitions  $[B_1, B_2]$  where  $\{1\} \in B_1$  and where one of the  $B_i$  contains a hyperedge of  $H$ , there are  $n + \binom{n}{2}$  elements in  $\Delta_0(H)$ . So, the dimension of  $H C_0(\Delta(H)) = \binom{n}{1}$ . It is clear that the dimension of  $H C_{-1}(\Delta(H)) = \binom{n}{0}$ .  $\square$

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